**Problem 1.** **(1 point)** `METUNCC/Applied_Math/least-squares/normal_eqn_4x3.pg`

Write the normal equation finding the best approximate solution to

$$\begin{bmatrix} -2 & -2 & 1 \\ -1 & -3 & -2 \\ 2 & -2 & -1 \\ 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ -2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \_\_ \\ \_\_ \\ \_\_ \end{bmatrix}$$

**Problem 2.** **(1 point)** `METUNCC/Applied_Math/least-squares/line.pg`

By using the method of least squares, find the best line through the points:
$(-2,-1),\ (1,0),\ (2,2),\ (0,-3).$

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**Step 1.** The general equation of a line is $c_0 + c_1 x = y$. Plugging the data points into this formula gives a matrix equation $\mathrm{A}\mathbf{c} = \mathbf{y}$.

$$\begin{bmatrix} \_\_ & \_\_ \\ \_\_ & \_\_ \\ \_\_ & \_\_ \\ \_\_ & \_\_ \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \end{bmatrix}$$

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**Step 2.** The matrix equation $\mathrm{A}\mathbf{c} = \mathbf{y}$ (probably) has no solution, so instead we use the **normal equation** $\mathrm{A}^{\mathrm{T}}\mathrm{A}\,\hat{\mathbf{c}} = \mathrm{A}^{\mathrm{T}}\mathbf{y}$

$$\mathrm{A}^{\mathrm{T}}\mathrm{A} = \begin{bmatrix} \_\_ & \_\_ \\ \_\_ & \_\_ \end{bmatrix}$$

$$\mathrm{A}^{\mathrm{T}}\mathbf{y} = \begin{bmatrix} \_\_ \\ \_\_ \end{bmatrix}$$

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**Step 3.** Solving the normal equation gives the answer

$$\hat{\mathbf{c}} = \begin{bmatrix} \_\_ \\ \_\_ \end{bmatrix}$$

which corresponds to the formula

$y =$ ____________

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**Analysis.** Compute the predicted $y$ values: $\hat{\mathbf{y}} = \mathrm{A}\hat{\mathbf{c}}$.

$$\hat{\mathbf{y}} = \begin{bmatrix} \_\_\_\_ \\ \_\_\_ \\ \_\_\_ \\ \_\_\_ \end{bmatrix}$$

Compute the error vector: $\mathbf{e} = \mathbf{y} - \hat{\mathbf{y}}$.

$$\mathbf{e} = \begin{bmatrix} \_\_\_\_ \\ \_\_\_ \\ \_\_\_ \\ \_\_\_ \end{bmatrix}$$

Compute the total error: $\text{SSE} = e_1^2 + e_2^2 + e_3^2 + e_4^2$.

$\text{SSE} =$ ____________

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**Problem 3.** **(1 point)** `METUNCC/Applied_Math/least-squares/QR_Compute-3x3.pg`

Compute a scaled QR decomposition of the matrix $A = \begin{bmatrix} -2 & -1 & -13 \\ 6 & -11 & 7 \\ -4 & 12 & 3 \end{bmatrix}$

$$A = \begin{bmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{bmatrix} \begin{bmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{bmatrix}$$

**Problem 4.** **(1 point)** `METUNCC/Applied_Math/least-squares/QR_Divide-3x3.pg`

In this problem you will use scaled $QR$ decomposition to divide

$$\begin{bmatrix} -1 & 6 & -2 \\ -1 & 6 & 2 \\ 4 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ -6 \\ -11 \end{bmatrix}$$

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**Step 1.** Divide by $Q$.

Use the fact that the columns of $Q$ are orthogonal to solve

$$\begin{bmatrix} -1 & 6 & -2 \\ -1 & 6 & 2 \\ 4 & 3 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -2 \\ -6 \\ -11 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \_\_\_ \\ \_\_\_ \\ \_\_\_ \end{bmatrix}$$

*Hint: All answers should simplify to be integers.*

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**Step 2.** Divide by $R$.

Use back-substitution to solve

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \_\_\_ \\ \_\_\_ \\ \_\_\_ \end{bmatrix}$$

**Problem 5.** **(1 point)** `METUNCC/Applied_Math/least-squares/Amatrix.pg`

In this problem for experts, you will compute the matrix $A$ used to make least square estimates in three exotic situations.

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**(A)** Construct the matrix equation used to find the least squares best fit of the data
$(3,\ 1),\quad (-3,\ -2),\quad (2,\ 2),\quad (-2,\ 0).$
to the formula $c_0\frac{1}{x}+c_1+c_2x=y$.

$$\begin{bmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \end{bmatrix}$$

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**(B)** Construct the matrix equation used to find the least squares best fit of the data
$(1,\ 1,\ 2),\quad (-3,\ -3,\ 3),\quad (3,\ -1,\ -1),\quad (-3,\ -3,\ 3).$
to the formula $c_0x+c_1y+c_2xy=z$.

$$\begin{bmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \_\_ \\ \_\_ \\ \_\_ \\ \_\_ \end{bmatrix}$$

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**(C)** Construct the matrix equation used to find the least squares best fit of the data $(0,\ -3),\quad (\frac{\pi}{6},\ 3),\quad (\frac{\pi}{3},\ -3)$
to the formula $c_0\sin(x)+c_1\cos(x)=y$

$$\begin{bmatrix} \_\_ & \_\_ \\ \_\_ & \_\_ \\ \_\_ & \_\_ \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} \_\_ \\ \_\_ \\ \_\_ \end{bmatrix}$$

**Problem 6.** **(1 point)** `METUNCC/Applied_Math/least-squares/QR_Normal-4x3.pg`

In this problem you will use scaled *QR* solve the normal equation for

$$\begin{bmatrix} -4 & 2 & 2 \\ 2 & -2 & 2 \\ 4 & 1 & 2 \\ -4 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ 4 \\ -4 \\ -4 \end{bmatrix}$$

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**Step 1.** Solve the normal equation with $Q$.

Use the fact that the columns of $Q$ are orthogonal to solve the normal equation for

$$\begin{bmatrix} -4 & 2 & 2 \\ 2 & -2 & 2 \\ 4 & 1 & 2 \\ -4 & -2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -11 \\ 4 \\ -4 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} \hat{a} \\ \hat{b} \\ \hat{c} \end{bmatrix} = \begin{bmatrix} \_\_ \\ \_\_ \\ \_\_ \end{bmatrix}$$

*Hint: All answers should simplify to be integers.*

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**Step 2.** Divide by $R$.

Use back-substitution to solve

$$\begin{bmatrix} 1 & 4 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \hat{a} \\ \hat{b} \\ \hat{c} \end{bmatrix}$$

$$\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \_\_ \\ \_\_ \\ \_\_ \end{bmatrix}$$